

ON THE LITTLEWOOD–OFFORD PROBLEM

YULIA S. ELISEEVA^{1,2}, ANDREI YU. ZAITSEV^{1,3}

ABSTRACT. The paper deals with studying a connection of the Littlewood–Offord problem with estimating the concentration functions of some symmetric infinitely divisible distributions. Some multivariate generalizations of results of Arak (1980) are given. They show a connection of the concentration function of the sum with the arithmetic structure of supports of distributions of independent random vectors for arbitrary distributions of summands.

Let X, X_1, \dots, X_n be independent identically distributed (i.i.d.) random variables. Let $a = (a_1, \dots, a_n)$, where $a_k = (a_{k1}, \dots, a_{kd}) \in \mathbf{R}^d$, $k = 1, \dots, n$. The concentration function of a \mathbf{R}^d -dimensional random vector Y with distribution $F = \mathcal{L}(Y)$ is defined by the equality

$$Q(F, \lambda) = \sup_{x \in \mathbf{R}^d} \mathbf{P}(Y \in x + \lambda B), \quad \lambda \geq 0,$$

where $B = \{x \in \mathbf{R}^n : \|x\| \leq 1/2\}$. In this paper we study the behavior of the concentration functions of the weighted sums $S_a = \sum_{k=1}^n X_k a_k$ with respect to the properties of vectors a_k .

Recently, interest in this subject has increased considerably in connection with the study of eigenvalues of random matrices (see, for instance, Friedland and Sodin [9], Nguyen and Vu [13], Rudelson and Vershynin [16], [17], Tao and Vu [18], [19], Vershynin [20]). For a detailed history of the problem we refer to a recent review of Nguen and Vu [14]. The authors of the above articles (see also Halász [10]) called this question the Littlewood–Offord problem, since, for the first time, this problem was considered in 1943 by Littlewood and Offord [12] in connection with the study of random polynomials. They considered a special case, where the coefficients $a_k \in \mathbf{R}$ are one-dimensional, and X takes values ± 1 with probabilities $1/2$.

Let us introduce some notation. In the sequel, let F_a denote the distribution of the sum S_a , E_y is the probability measure concentrated at a point y , and let G be the distribution of the random variable \tilde{X} , where $\tilde{X} = X_1 - X_2$ is the symmetrized random variable.

The symbol c will be used for absolute positive constants which may be different even in the same formulas.

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Writing $A \ll B$ means that $|A| \leq cB$. Also we will write $A \asymp B$, if $A \ll B$ and $B \ll A$. We will write $A \ll_d B$, if $|A| \leq c(d)B$, where $c(d) > 0$ depends on d only. Similarly, $A \asymp_d B$, if $A \ll_d B$ and $B \ll_d A$. The scalar product in \mathbf{R}^d will be denoted $\langle \cdot, \cdot \rangle$. Later $\lfloor x \rfloor$ is the largest integer k such that $k < x$. For $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ we will use the norms $\|x\|^2 = x_1^2 + \dots + x_n^2$ and $|x| = \max_j |x_j|$. We denote by $\widehat{F}(t)$, $t \in \mathbf{R}^d$, the characteristic function of d -dimensional distributions F .

Products and powers of measures will be understood in the convolution sense. While a distribution F is infinitely divisible, F^λ , $\lambda \geq 0$, is the infinitely divisible distribution with characteristic function $\widehat{F}^\lambda(t)$.

The elementary properties of concentration functions are well studied (see, for instance, [3], [11], [15]). It is known that

$$Q(F, \mu) \ll_d (1 + \lfloor \mu/\lambda \rfloor)^d Q(F, \lambda) \quad (1)$$

for any $\mu, \lambda > 0$. Hence,

$$Q(F, c\lambda) \asymp_d Q(F, \lambda). \quad (2)$$

Let us formulate a generalization of the classical Esséen inequality [7] to the multivariate case ([8], see also [11]):

Lemma 1. *Let $\tau > 0$ and let F be a d -dimensional probability distribution. Then*

$$Q(F, \tau) \ll_d \tau^d \int_{|t| \leq 1/\tau} |\widehat{F}(t)| dt. \quad (3)$$

In the general case $Q(F, \lambda)$ cannot be estimated from below by the right hand side of inequality (3). However, if we assume additionally that the distribution F is symmetric and its characteristic function is non-negative for all $t \in \mathbf{R}$, then we have the lower bound:

$$Q(F, \tau) \gg_d \tau^d \int_{|t| \leq 1/\tau} |\widehat{F}(t)| dt, \quad (4)$$

and, therefore,

$$Q(F, \tau) \asymp_d \tau^d \int_{|t| \leq 1/\tau} |\widehat{F}(t)| dt, \quad (5)$$

(see [1] or [3], Lemma 1.5 of Chapter II for $d = 1$). In the multivariate case relations (4) and (5) were obtained by Zaitsev [21], see also Eliseeva [4]. Just the use of relation (5) allows us to simplify the arguments of Friedland and Sodin [9], Rudelson and Vershynin [17] and Vershynin [20] which were applied to Littlewood–Offord problem (see [4], [5] and [6]).

The main result of this paper is a general inequality which reduces the estimation of concentration functions in the Littlewood–Offord problem to the estimation of concentration functions of some infinitely divisible distributions. This result is formulated in Theorem 1.

For $z \in \mathbf{R}$, introduce the distribution H_z with the characteristic function

$$\widehat{H}_z(t) = \exp \left(-\frac{1}{2} \sum_{k=1}^n (1 - \cos(\langle t, a_k \rangle z)) \right). \quad (6)$$

It depends on the vector a . It is clear that H_z is a symmetric infinitely divisible distribution. Therefore, its characteristic function is positive for all $t \in \mathbf{R}^d$.

Theorem 1. *Let V be an arbitrary d -dimensional Borel measure such that $\lambda = V\{\mathbf{R}\} > 0$, and $V \leq G$, that is, $V\{B\} \leq G\{B\}$, for any Borel set B . Then, for any $\varepsilon > 0$ and $\tau > 0$, we have*

$$Q(F_a, \tau) \ll_d Q(H_1^\lambda, \varepsilon) \exp \left(d \int_{z \in \mathbf{R}} \log(1 + \lfloor \tau(\varepsilon|z|)^{-1} \rfloor) F\{dz\} \right), \quad (7)$$

where $F = \lambda^{-1}V$.

Note that $\log(1 + \lfloor \tau(\varepsilon|z|)^{-1} \rfloor) = 0$, for $|z| \geq \tau/\varepsilon$. Therefore, the integration in (7) is taken, in fact, over the set $\{z : |z| < \tau/\varepsilon\}$ only.

Corollary 1. *Let $\delta > 0$ and*

$$p(\delta) = G\{\{z : |z| \geq \delta\}\} > 0. \quad (8)$$

Then, for any $\varepsilon, \tau > 0$, we have

$$Q(F_a, \tau) \ll_d e^\Delta Q(H_1^{p(\delta)}, \varepsilon), \quad (9)$$

where

$$\Delta = \Delta(\tau, \varepsilon, \delta) = \frac{d}{p(\delta)} \int_{|z| \geq \delta} \log(1 + \lfloor \tau(\varepsilon|z|)^{-1} \rfloor) G\{dz\}. \quad (10)$$

In particular, choosing $\delta = \tau/\varepsilon$, we get

Corollary 2. *For any $\varepsilon, \tau > 0$, we have*

$$Q(F_a, \tau) \ll_d Q(H_1^{p(\tau/\varepsilon)}, \varepsilon). \quad (11)$$

Just the statement of Corollary 2 (usually for $\tau = \varepsilon$) is actually the starting point of almost all recent studies on the Littlewood–Offord problem (see, for instance, [9], [10], [13], [16], [17] and [20]). More precisely, with the help of Lemma 1 or its analogs, the authors of the above-mentioned papers have obtained estimates of the type

$$Q(F_a, \tau) \ll_d \sup_{z \geq \tau/\varepsilon} \tau^d \int_{|t| \leq 1/\tau} \widehat{H}_z^{p(\tau/\varepsilon)}(t) dt. \quad (12)$$

The fact that (1) and (5) imply that

$$\begin{aligned} \sup_{z \geq \tau/\varepsilon} \tau^d \int_{|t| \leq 1/\tau} \widehat{H}_z^{p(\tau/\varepsilon)}(t) dt &\asymp_d \sup_{z \geq \tau/\varepsilon} Q(H_z^{p(\tau/\varepsilon)}, \tau) \\ &= \sup_{z \geq \tau/\varepsilon} Q(H_1^{p(\tau/\varepsilon)}, \tau/z) = Q(H_1^{p(\tau/\varepsilon)}, \varepsilon), \end{aligned} \quad (13)$$

remained apparently unnoticed by the authors of these papers that significantly hampered further evaluation of the right-hand side of inequality (12).

Choosing V so that

$$V\{dz\} = \left(\max \{1, \log(1 + \lfloor \tau(\varepsilon|z|)^{-1} \rfloor)\} \right)^{-1} G\{dz\}, \quad (14)$$

we obtain

Corollary 3. *For any $\varepsilon, \tau > 0$, we have*

$$Q(F_a, \tau) \ll_d Q(H_1^\lambda, \varepsilon) \exp(d\lambda^{-1} G\{|z| < \tau/\varepsilon\}), \quad (15)$$

where

$$\lambda = \lambda(G, \tau/\varepsilon) = V\{\mathbf{R}\} = \int_{z \in \mathbf{R}} \left(\max \{1, \log(1 + \lfloor \tau(\varepsilon|z|)^{-1} \rfloor)\} \right)^{-1} G\{dz\}. \quad (16)$$

In Corollaries 1–3 we choose the measure V in the form $V\{dz\} = f(z)G\{dz\}$ with $0 \leq f(z) \leq 1$. It is not clear what choice of f is optimal. This depends on a and G .

Choosing the optimal function f , minimizing the right-hand sides of inequalities (9), (11) and (15), is a difficult problem. It is clear that its solution depends on a and G . Certainly, it is sufficient to consider non-decreasing functions f only.

For a fixed ε , an increase of λ implies a decrease of $Q(H_1^\lambda, \varepsilon)$. Theorem 1 may be applied for $V = G$. Then $\lambda = 1$. This is the maximal possible value of λ . However, the integral in the right-hand side of (7) may be in this case infinite. In particular, it diverges if the distribution G has a nonzero atom at zero. This atom in any case should be excluded in constructing the measure V , if we expect to get a meaningful bound for $Q(F_a, \tau)$. For a fixed measure V , decreasing ε implies a decrease of $Q(H_1^\lambda, \varepsilon)$, but an increase of the integral in the right-hand side of inequality (7).

In Corollary 3 we used the measure V , defined in (14) so that the integral in the right-hand side of inequality (7) would converge always, no matter what is the measure G .

The proof of Theorem 1 is based on elementary properties of concentration functions, it will be given below. Note that H_1^λ is an infinitely divisible distribution with the Lévy

spectral measure $M_\lambda = \frac{\lambda}{4} M^*$, where $M^* = \sum_{k=1}^n (E_{a_k} + E_{-a_k})$. It is clear that the assertions

of Theorem 1 and Corollaries 1–3 reduce the Littlewood–Offord problem to the study of the measure M^* , uniquely corresponding to the vector a . In fact, almost all the results obtained when solving this problem, are formulated in terms of the coefficients a_j or, equivalently, in terms of the properties of the measure M^* . Sometimes this leads to a loss of information on the distribution of the random variable X , which can help in obtaining more precise estimates. In particular, if $\mathcal{L}(X)$ is the standard normal distribution, then F_a is a Gaussian distribution with zero mean and covariance operator which can be easily calculated. Thus, there are situations in which it is possible to obtain estimates for $Q(F_a, \tau)$ that do not follow from the results formulated in terms of the measure M^* .

Note that using the results of Arak [1], [2] (see also [3]) one could derive from Theorem 1 estimates similar to estimates of concentration functions in the Littlewood–Offord problem,

which were obtained in a recent paper of Nguyen and Vu [13] (see also [14]). A detailed discussion of this fact is presented in a joint paper of the authors and Friedrich Götze which is preparing for the publication. In the same paper there is a proof of multidimensional analogs of some results of Arak [1]. In Theorems 2 and 3 below, we provide without proof the formulations of these results which demonstrates a relation between the order of smallness of the concentration function of the sum and the arithmetic structure of the supports of distributions of independent random vectors for *arbitrary* distributions of summands, in contrast to the results of [9], [13], [16]–[20], in which a similar relationship was found in a particular case of summands with the distributions arising in the Littlewood–Offord problem.

We need some notation. Let \mathbf{Z}_+ be the set non-negative integers. For any $r \in \mathbf{Z}_+$ and $u = (u_1, \dots, u_r) \in (\mathbf{R}^d)^r$, $u_j \in \mathbf{R}^d$, $j = 1, \dots, r$, introduce the sets

$$K_1(u) = \left\{ \sum_{j=1}^r n_j u_j : n_j \in \{-1, 0, 1\} \text{ for } j = 1, \dots, r \right\}. \quad (17)$$

We denote by $[B]_\tau$ the closed τ -neighborhood of a set B in the sense of the norm $|\cdot|$.

Theorem 2. *Let $\tau \geq 0$ and let F_j , $j = 1, \dots, n$, be d -dimensional probability distributions. Denote $\gamma = Q\left(\prod_{j=1}^n F_j, \tau\right)$. Then there exist $r \in \mathbf{Z}_+$ and vectors $u_1, \dots, u_r; x_1, \dots, x_r \in \mathbf{R}^d$ such that*

$$r \ll_d |\log \rho| + 1, \quad (18)$$

and

$$\sum_{j=1}^n F_j\{\mathbf{R}^d \setminus [K_1(u)]_\tau + x_j\} \ll_d (|\log \rho| + 1)^3, \quad (19)$$

where $u = (u_1, \dots, u_r) \in (\mathbf{R}^d)^r$, and the set $K_1(u)$ is defined in (17).

Theorem 3. *Let D be a d -dimensional infinitely divisible distribution with characteristic function of the form $\exp\{\alpha(\widehat{M}(t) - 1)\}$, $t \in \mathbf{R}^d$, where $\alpha > 0$ and M is a probability distribution. Let $\tau \geq 0$ and $\gamma = Q(D, \tau)$. Then there exist $r \in \mathbf{Z}_+$ and vectors $u_1, \dots, u_r \in \mathbf{R}^d$ such that*

$$r \ll_d |\log \gamma| + 1, \quad (20)$$

and

$$\alpha M\{\mathbf{R}^d \setminus [K_1(u)]_\tau\} \ll_d (|\log \gamma| + 1)^3, \quad (21)$$

where $u = (u_1, \dots, u_r) \in (\mathbf{R}^d)^r$.

Proof of Theorem 1. Let us show that, for arbitrary probability distribution F and $\lambda, T > 0$,

$$\begin{aligned} & \log \int_{|t| \leq T} \exp \left(-\frac{1}{2} \sum_{k=1}^n \int_{z \in \mathbf{R}} (1 - \cos(\langle t, a_k \rangle z)) \lambda F\{dz\} \right) dt \\ & \leq \int_{z \in \mathbf{R}} \left(\log \int_{|t| \leq T} \exp \left(-\frac{\lambda}{2} \sum_{k=1}^n (1 - \cos(\langle t, a_k \rangle z)) \right) dt \right) F\{dz\} \\ & = \int_{z \in \mathbf{R}} \left(\log \int_{|t| \leq T} \widehat{H}_z^\lambda(t) dt \right) F\{dz\}. \quad (22) \end{aligned}$$

It suffices to prove (22) for discrete distributions $F = \sum_{j=1}^{\infty} p_j E_{z_j}$, where $0 \leq p_j \leq 1$, $z_j \in \mathbf{R}$, $\sum_{j=1}^{\infty} p_j = 1$. Applying in this case the Hölder inequality, we have

$$\begin{aligned} & \int_{|t| \leq T} \exp \left(-\frac{1}{2} \sum_{k=1}^n \int_{z \in \mathbf{R}} (1 - \cos(\langle t, a_k \rangle z)) \lambda F\{dz\} \right) dt \\ & = \int_{|t| \leq T} \exp \left(-\frac{\lambda}{2} \sum_{j=1}^{\infty} p_j \sum_{k=1}^n (1 - \cos(\langle t, a_k \rangle z_j)) \right) dt \\ & \leq \prod_{j=1}^{\infty} \left(\int_{|t| \leq T} \exp \left(-\frac{\lambda}{2} \sum_{k=1}^n (1 - \cos(\langle t, a_k \rangle z_j)) \right) dt \right)^{p_j}. \quad (23) \end{aligned}$$

Taking the logarithms of the left and right-hand sides of (23), we get (22). In general case we can approximate the distribution F by discrete distributions in the sense of weak convergence and to pass to the limit. We use that the weak convergence of probability distributions is equivalent to the convergence of characteristic functions which is uniform on bounded sets. Moreover, the weak convergence of symmetric infinitely divisible distributions is equivalent to the weak convergence of the corresponding spectral measure. Note also that the integrals $\int_{|t| \leq T}$ may be replaced in (22) by the integrals $\int_{t \in B}$ over an arbitrary Borel set B .

Since for characteristic function $\widehat{W}(t)$ of a random vector Y , we have

$$|\widehat{W}(t)|^2 = \mathbf{E} \exp(i \langle t, \widetilde{Y} \rangle) = \mathbf{E} \cos(\langle t, \widetilde{Y} \rangle),$$

where \widetilde{Y} is the corresponding symmetrized random vector, then

$$|\widehat{W}(t)| \leq \exp \left(-\frac{1}{2} (1 - |\widehat{W}(t)|^2) \right) = \exp \left(-\frac{1}{2} \mathbf{E} (1 - \cos(\langle t, \widetilde{Y} \rangle)) \right). \quad (24)$$

According to Theorem 1 and relations $V = \lambda F \leq G$, (22) and (24), we have

$$\begin{aligned}
Q(F_a, \tau) &\ll_d \tau^d \int_{\tau|t| \leq 1} |\widehat{F}_a(t)| dt \\
&\ll_d \tau^d \int_{\tau|t| \leq 1} \exp \left(-\frac{1}{2} \sum_{k=1}^n \mathbf{E} (1 - \cos(\langle t, a_k \rangle \tilde{X})) \right) dt \\
&= \tau^d \int_{\tau|t| \leq 1} \exp \left(-\frac{1}{2} \sum_{k=1}^n \int_{z \in \mathbf{R}} (1 - \cos(\langle t, a_k \rangle z)) G\{dz\} \right) dt \\
&\leq \tau^d \int_{\tau|t| \leq 1} \exp \left(-\frac{1}{2} \sum_{k=1}^n \int_{z \in \mathbf{R}} (1 - \cos(\langle t, a_k \rangle z)) \lambda F\{dz\} \right) dt \\
&\leq \exp \left(\int_{z \in \mathbf{R}} \log \left(\tau^d \int_{\tau|t| \leq 1} \widehat{H}_z^\lambda(t) dt \right) F\{dz\} \right). \tag{25}
\end{aligned}$$

Using (1) and (5), we have

$$\begin{aligned}
\tau^d \int_{\tau|t| \leq 1} \widehat{H}_z^\lambda(t) dt &\asymp_d Q(H_z^\lambda, \tau) = Q(H_1^\lambda, \tau|z|^{-1}) \\
&\leq (1 + \lfloor \tau(\varepsilon|z|)^{-1} \rfloor)^d Q(H_1^\lambda, \varepsilon). \tag{26}
\end{aligned}$$

Substituting this estimate into (25), we obtain (7). \square

REFERENCES

- [1] T. V. Arak, *On the approximation by the accompanying laws of n -fold convolutions of distributions with nonnegative characteristic functions.* — Theory Probab. Appl. **25**, 2 (1980), 221–243.
- [2] T. V. Arak, *On the convergence rate in Kolmogorov’s uniform limit theorem. I.* — Theory Probab. Appl. **26**, 2 (1981), 219–239.
- [3] T. V. Arak, A. Yu. Zaitsev, *Uniform limit theorems for sums of independent random variables.* — Proc. Steklov Inst. Math. **174** (1988), 222 p.
- [4] Yu. S. Eliseeva, *Multivariate estimates for the concentration functions of weighted sums of independent identically distributed random variables.* — Zap. Nauchn. Semin. POMI **412** (2013), 121–137 (in Russian), English version: arXiv:1303.4005.
- [5] Yu. S. Eliseeva, F. Götze, A. Yu. Zaitsev, *Estimates for the concentration functions in the Littlewood–Offord problem.* — Zap. Nauchn. Semin. POMI **420** (2013), 50–69 (in Russian), English version: arXiv:1203.6763.
- [6] Yu. S. Eliseeva, A. Yu. Zaitsev, *Estimates for the concentration functions of weighted sums of independent random variables.* — Theory Probab. Appl. **57**, 4 (2013), 670–678.
- [7] C.-G. Esséen, *On the Kolmogorov–Rogozin inequality for the concentration function.* — Z. Wahrscheinlichkeitstheorie Verw. Geb. **5** (1966), 210–216.
- [8] C.-G. Esséen, *On the concentration function of a sum of independent random variables.* — Z. Wahrscheinlichkeitstheorie Verw. Geb. **9** (1968), 290–308.
- [9] O. Friedland, S. Sodin, *Bounds on the concentration function in terms of Diophantine approximation.* — C. R. Math. Acad. Sci. Paris **345** (2007), 513–518.

- [10] G. Halász, *Estimates for the concentration function of combinatorial number theory and probability*. — Periodica Mathematica Hungarica **8** (1977), 197–211.
- [11] W. Hengartner, R. Theodorescu, *Concentration Functions*. Academic Press, New York, 1973.
- [12] J. E. Littlewood, A. C. Offord, *On the number of real roots of a random algebraic equation*. — Rec. Math. [Mat. Sbornik] N.S. **12** (1943), 277–286.
- [13] H. Nguyen, V. Vu, *Optimal inverse Littlewood–Offord theorems*. — Adv. Math. **226** (2011), 5298–5319.
- [14] H. Nguyen, V. Vu, *Small probabilities, inverse theorems and applications*. — arXiv:1301.0019 (2013).
- [15] V. V. Petrov, *Sums of Independent Random Variables*. Nauka, Moscow, 1972.
- [16] M. Rudelson, R. Vershynin, *The Littlewood–Offord problem and invertibility of random matrices*. — Adv. Math. **218** (2008), 600–633.
- [17] M. Rudelson, R. Vershynin, *The smallest singular value of a random rectangular matrix*. — Comm. Pure Appl. Math. **62** (2009), 1707–1739.
- [18] T. Tao, V. Vu, *Inverse Littlewood–Offord theorems and the condition number of random discrete matrices*. — Ann. Math. **169** (2009), 595–632.
- [19] T. Tao, V. Vu, *From the Littlewood–Offord problem to the circular law: universality of the spectral distribution of random matrices*. — Bull. Amer. Math. Soc. **46** (2009), 377–396.
- [20] R. Vershynin, *Invertibility of symmetric random matrices*. — Random Structures Algorithms **44** (2014), 135–182.
- [21] A. Yu. Zaitsev, *Multidimensional generalized method of triangular functions*. — J. Soviet Math. **43**, 6 (1988), 2797–2810.

E-mail address: pochta106@yandex.ru

ST. PETERSBURG STATE UNIVERSITY
AND LABORATORY OF CHEBYSHEV IN ST. PETERSBURG STATE UNIVERSITY

E-mail address: zaitsev@pdmi.ras.ru

ST. PETERSBURG DEPARTMENT OF STEKLOV MATHEMATICAL INSTITUTE
FONTANKA 27, ST. PETERSBURG 191023, RUSSIA
AND ST. PETERSBURG STATE UNIVERSITY